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LETTER TO THE EDITOR

New integrable model of correlated electrons with off-diagonal long-range order from $so(5)$ symmetry

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Abstract. We present a new integrable model for correlated electrons which is based on $so(5)$ symmetry. By using an η-pairing realization we construct eigenstates of the Hamiltonian with off-diagonal long-range order. It is also shown that these states lie in the ground state sector. We exactly solve the model on a one-dimensional lattice by the Bethe ansatz.

The study of models of correlated electrons is a significant tool in the theory of condensed matter physics. On a one-dimensional lattice there are several known models which are exactly solvable by Bethe ansatz methods. The most famous of these is the Hubbard model whose solution was obtained by Lieb and Wu [1]. Another well known example is the t–J model, the strong-coupling limit of the Hubbard model, which was in fact shown to be integrable at the supersymmetric point [2] through use of the quantum inverse scattering method (QISM) [3]. In this formalism the Hamiltonian of the model is derived from a solution of the Yang–Baxter equation, hereafter referred to as an R-matrix, which provides a systematic method to obtain higher-order conservation laws that guarantee integrability. An important aspect of the integrable coupling of the t–J model is that the R-matrix is invariant with respect to the Lie superalgebra $gl(2|1)$. For the case of the Hubbard model the symmetry algebra has been identified as $so(4)$ [4].

A further important integrable correlated electron model was proposed and solved through the algebraic Bethe ansatz method by Essler *et al* [5]. This model generalizes the Hubbard model with the addition of correlated hopping and pair hopping terms and is constructed from an R-matrix invariant with respect to the Lie superalgebra $gl(2|2)$. Another direction of generalization was given by Bracken *et al* [6] using the R-matrix obtained from the one-parameter family of four-dimensional representations of $gl(2|1)$. The resulting model, known as the supersymmetric U model has also been solved and analysed by Bethe ansatz techniques [7]. Other types of Hubbard model generalizations can be found in [8]. In all the above examples the underlying symmetry has crucial consequences for the multiplet structure of the models providing insight into the ground state and elementary excitations.

Recently, it has been proposed that the antiferromagnetic and superconducting phases of high-T_c cuprate compounds are unified by an approximate $so(5)$ symmetry [9]. Considerable support for this proposal came from numerical investigations in models for high-T_c materials. In particular, it was shown that the low-energy excitations can be classified in terms of an $so(5)$

symmetry multiplet structure [10, 11]. Subsequently, extended Hubbard models related with an $so(5)$ symmetry have been introduced and analysed in detail [12].

To our knowledge, no integrable correlated electron model associated with an $so(5)$ symmetry has been proposed nor exactly solved. In this paper we construct such a correlated electron model which is exactly solved in one dimension by the Bethe ansatz. The integrability of our Hamiltonian comes from the fact that it is derived from an $so(5)$-invariant R-matrix which satisfies the quantum Yang–Baxter equation. Eigenstates of this Hamiltonian exhibiting off-diagonal long-range order (ODLRO) can be constructed through an η-pairing mechanism. We also argue that these states lie in the ground state sector, which is a prerequisite for superconductivity.

The Hamiltonian of this model is given by

$$H = \sum_{i=1}^{L-1} h_{i,i+1} + h_{L,1} + \mu N + BS^z \tag{1}$$

where

$$\begin{aligned} h_{i,j} = -\sum_{\sigma=\uparrow,\downarrow} & c^\dagger_{i\sigma} c_{j\sigma}(3 - 2n_{i,-\sigma} - 4n_{j,-\sigma}) + \text{h.c.} \\ & -2(c^\dagger_{i\uparrow} c^\dagger_{i\downarrow} c_{j\downarrow} c_{j\uparrow} + \text{h.c.}) - 4(S^x_i S^x_j + S^y_i S^y_j + 3S^z_i S^z_j) \\ & -3(n_i + n_j) - 3n_i n_j - 4[(\tfrac{1}{4}(n_i - n_j)^2 - (S^z_i - S^z_j)^2]^2. \end{aligned} \tag{2}$$

Above, $c_{i\sigma}$, $c^\dagger_{i\sigma}$ are annihilation and creation operators for electrons of spin σ, the $\vec{S}_i$ spin matrices and the $n_{i\uparrow}$, $n_{i\downarrow}$ occupation numbers of electrons at lattice site i. The number of lattice sites is L, $S^z = \sum_{i=1}^L S^z_i$ is the magnetization and $N = \sum_{i=1}^L (n_{i\uparrow} + n_{i\downarrow})$ is the number of electrons. This Hamiltonian exhibits correlated electron hoppings, pair hoppings, XXZ-type interaction, chemical potential, nearest-neighbouring Coulomb interaction, and the last term characterizes interactions favouring antiferromagnetism. The energy levels of the model are

$$E = -\sum_j \frac{12}{4u_j^2 - 1} + \mu N + BS^z \tag{3}$$

where the u_j are solutions of the Bethe ansatz equations

$$\begin{aligned} \left(\frac{u_i + \frac{1}{2}}{u_i - \frac{1}{2}}\right)^L &= -(-1)^{M_1} \prod_{j\neq i}^{M_1} \frac{u_i - u_j + 1}{u_i - u_j - 1} \prod_k^{M_2} \frac{u_i - \overline{u_k} - 1}{u_i - \overline{u_k} + 1} \qquad i = 1, \ldots, M_1 \\ 1 &= \prod_j^{M_1} \frac{u_j - \overline{u_i} - 1}{u_j - \overline{u_i} + 1} \prod_{k\neq i}^{M_2} \frac{\overline{u_k} - \overline{u_i} + 2}{\overline{u_k} - \overline{u_i} - 2} \qquad j = 1, \ldots, M_2 \end{aligned} \tag{4}$$

where $M_1 = 2L - N$ and $M_2 = L - N_\uparrow$. Integrability of this model will be established through the QISM. The energy eigenvalues as well as the Bethe ansatz equations are obtained through the analytic Bethe ansatz [13]. The key ingredient to both of these methods is the following R-matrix [14]:

$$R(u) = \sum_{i,j} (u(u-3)e^i_i \otimes e^j_j + (3-u)e^i_j \otimes e^j_i + (-1)^{i+j} u e^i_j \otimes e^{\bar{i}}_{\bar{j}}) \tag{5}$$

which satisfies the Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{6}$$

Above, the matrices e^i_j have entries $(e^i_j)^k_l = \delta_{ik}\delta_{jl}$, the indices range from 1 to 4 and $\bar{i} = 5 - i$. This R-matrix possesses the properties of

$$\text{unitarity} \qquad R(u)R(-u) = (u^2-1)(u^2-9)I \otimes I \tag{7}$$

$$\text{crossing symmetry} \qquad R^{t_1}(u) = -A_1 R(3-u) A_1 \tag{8}$$

where t_1 denotes transposition in the first space and $A = \sum_i (-1)^i e^i_{\bar{i}}$.

The solution (5) is invariant with respect to the Lie algebra $so(5) \cong sp(4)$ which has ten generators

$$a^i_j = e^i_j - (-1)^{i+j} e^{\bar{j}}_{\bar{i}} = -(-1)^{i+j} a^{\bar{j}}_{\bar{i}} \tag{9}$$

satisfying the commutation relations

$$[a^i_j, a^k_l] = \delta^k_j a^i_l - \delta^i_l a^k_j + (-1)^{i+j} \delta^{\bar{j}}_l a^k_{\bar{i}} - (-1)^{i+j} \delta^k_{\bar{i}} a^{\bar{j}}_l.$$

In order to build an electronic model we first need to put $\boldsymbol{Z}_2$ grading in the R-matrix. This is achieved by a redefinition of the matrix elements through

$$R(u)^{ij}_{kl} \to (-1)^{[i][j]+[k][j]+[k][l]} R(u)^{ij}_{kl} \tag{10}$$

where we choose the parities to be

$$[1] = [4] = 0 \qquad [2] = [3] = 1 \qquad \text{and} \qquad [e^i_j] = [i] + [j].$$

Such a matrix satisfies the $\boldsymbol{Z}_2$ graded Yang–Baxter where the multiplication of tensor products of matrices is governed by

$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]} ac \otimes bd$$

in equation (6). Following the QISM, we may construct the transfer matrix

$$\tau(u) = \mathrm{str}_0(R_{0L}(u) R_{0L-1}(u) \ldots, R_{02}(u) R_{01}(u)) \tag{11}$$

where str_0 is the supertrace over the zeroth space. From the Yang–Baxter algebra it follows that the transfer matrices $\tau(u)$ form a commuting family and the associated Hamiltonian (1) with $\mu = 0$ and $B = 0$ can be obtained from

$$H = \tau(u)^{-1} \frac{\mathrm{d}}{\mathrm{d}u} \tau(u) \bigg|_{u=0}$$

where, in view of the grading, we have used the following identification:

$$|1\rangle \equiv |\uparrow\downarrow\rangle \qquad |2\rangle \equiv |\uparrow\rangle \qquad |3\rangle \equiv |\downarrow\rangle \qquad |4\rangle \equiv |0\rangle.$$

In terms of the fermion operators, the $so(5)$ generators (9) can be written as

$$\begin{array}{lllll} a^1_1 = n - 1 & a^2_2 = 2S^z & a^1_2 = c^\dagger_- & a^2_1 = c_- & a^1_3 = -c^\dagger_+ \\ a^3_1 = -c_+ & a^1_4 = 2c^\dagger_- c^\dagger_+ & a^4_1 = 2c_+ c_- & a^2_3 = 2S^+ & a^3_2 = 2S^-. \end{array} \tag{12}$$

On the two-fold tensor product space these generators act according to the co-product

$$\begin{array}{ll} \Delta(a^i_j) = a^i_j \otimes I + (-1)^n \otimes a^i_j & \text{for} \quad a^i_j = a^1_2, a^2_1, a^1_3, a^3_1 \\ \Delta(a^i_j) = a^i_j \otimes I + I \otimes a^i_j & \text{otherwise} \end{array} \tag{13}$$

which extends to the L-fold tensor space co-associatively. Each of the local Hamiltonians $h_{i,i+1}(2)$ are $so(5)$-invariant. However due to the non-cocommutativity of the co-product the $h_{L,1}$ term breaks the $so(5)$ symmetry of the global Hamiltonian (1). In spite of this, an $so(4)$ symmetry is preserved comprising of an $so(3)$ spin realization and an additional $so(3)$ η-pairing realization. For this reason we can add arbitrary chemical potential and magnetic field terms to the Hamiltonian which do not violate the integrability.

The presence of the η-pairing realization

$$\eta = \sum_{j=1}^{L} c_{j,\uparrow} c_{j,\downarrow} \qquad \eta^\dagger = \sum_{j=1}^{L} c^\dagger_{j,\downarrow} c^\dagger_{j,\uparrow} \qquad \eta^z = \sum_{j=1}^{L} \tfrac{1}{2}(n_j - 1) \tag{14}$$

which can also be expressed in terms of the subalgebra generated by $\{a_1^1, a_4^1, a_1^4\}$, allows a large number of states to be constructed exhibiting ODLRO [15]. Hereafter, we treat the case $\mu = 0$, $B = 0$. One can verify that $H|0\rangle = 0$ where $|0\rangle$ denotes the vacuum state. Thus the $2\mathcal{N}$ electron states

$$|\Psi_\mathcal{N}\rangle = (\eta^\dagger)^\mathcal{N}|0\rangle \tag{15}$$

are eigenstates of the global Hamiltonian with zero energy. These states are well known to possess ODLRO; that is

$$\lim_{|l-j|\to\infty} \frac{\langle\Psi_\mathcal{N}|c^\dagger_{j,\downarrow} c^\dagger_{j,\uparrow} c_{l,\uparrow} c_{l,\downarrow}|\Psi_\mathcal{N}\rangle}{\langle\Psi_\mathcal{N}|\Psi_\mathcal{N}\rangle} = \frac{\mathcal{N}}{L}\left(1 - \frac{\mathcal{N}}{L}\right) \tag{16}$$

in the thermodynamic limit ($\mathcal{N}, L \to \infty, \mathcal{N}/L$ fixed). Since the Hamiltonian is Hermitian the ground state energy satisfies

$$E \geqslant LE_0$$

where E_0 is the minimum energy of the two-site Hamiltonian. For this model we can determine that $E_0 = 0$. It is thus concluded that the states (15) lie in the ground state sector.

The energy levels (3) are determined from the eigenvalues of the transfer matrix (11) which leads to a complicated expression that we will not give here. However, we mention that these eigenvalues are obtained through the analytic Bethe ansatz which exploits the properties of unitarity (7), crossing symmetry (8) and asymptotic behaviour of the R-matrix. As usual, the Bethe ansatz equations are derived by the requirement that the eigenvalues are analytic functions.

By a suitable change in the boundary conditions, one may recover a closed model with exact $so(5)$ symmetry. In this instance the Hamiltonian reads

$$H = \sum_{i=1}^{L-1} h_{i,i+1} + (-1)^{n_1(N-n_1)} h_{L,1} (-1)^{n_1(N-n_1)}$$

and the energies are still given by the expression (3) with $\mu = B = 0$ but now subject to the modified Bethe ansatz equations

$$\begin{aligned} \left(\frac{u_i + \frac{1}{2}}{u_i - \frac{1}{2}}\right)^L &= -\prod_{j\neq i}^{M_1} \frac{u_i - u_j + 1}{u_i - u_j - 1} \prod_{k}^{M_2} \frac{u_i - \overline{u_k} - 1}{u_i - \overline{u_k} + 1} \qquad i = 1, \ldots, M_1 \\ 1 &= \prod_{j}^{M_1} \frac{u_j - \overline{u_i} - 1}{u_j - \overline{u_i} + 1} \prod_{k\neq i}^{M_2} \frac{\overline{u_k} - \overline{u_i} + 2}{u_k - \overline{u_i} - 2} \qquad j = 1, \ldots, M_2. \end{aligned} \tag{17}$$

Integrability in this case follows from the general construction of [16]. Because of the presence of exact $so(5)$ symmetry for this model we can describe the ground state structure. For a chain of length L it is an $so(5)$ multiplet of dimension $\frac{1}{6}(L+3)(L+2)(L+1)$ (rank L symmetric representation) which contains states of all possible fillings $0 \leqslant N \leqslant 2L$ and magnetizations $-L \leqslant S^z \leqslant L$.

In conclusion, we have introduced a new integrable correlated electron model based on an $so(5)$ symmetry. The model was exactly solved through the Bethe ansatz and shown to have ground states exhibiting ODLRO.

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